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MECHANICS

[Numbers 307 and 308 were omitted in the May issue.]

307. Proposed by LAENAS G. WELD, Pullman, Illinois.

Four forces W, X, Y, and Z, concurrent in O, are in equilibrium. Prove that

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & \cos XOY & \cos XOZ \\ \cos XOY & 1 & \cos YOZ \\ \cos XOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_2 = \begin{vmatrix} 1 & \cos WOY & \cos WOZ \\ \cos WOY & 1 & \cos YOZ \\ \cos WOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}};$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos WOX & \cos WOZ \\ \cos WOX & 1 & \cos XOZ \\ \cos WOZ & \cos XOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_4 = \begin{vmatrix} 1 & \cos WOX & \cos WOY \\ \cos WOX & 1 & \cos XOY \\ \cos WOY & \cos XOY & 1 \end{vmatrix}^{\frac{1}{2}}.$$

308. Proposed by H. S. UHLER, Yale University.

Prove that when a ray of light passes obliquely through a prism in such a manner as to maintain a constant value for the total deviation of the projection of the ray on a principal section, the ray inside the prism generates a cone of elliptical right section. It is assumed that the prism is surrounded by a medium having a smaller index of refraction than the index of the material of the prism.

NUMBER THEORY.

234. Proposed by FRANK IRWIN, University of California.

Start with any number, for instance 89, and add to it the number obtained by reversing the order of its digits: $89 + 98 = 187$. Now perform the same operation on the result: $187 + 781 = 968$. If we continue in this way we arrive, after a certain number of operations, at a number which reads the same forwards and backwards (24 operations bring us to 8813200023188). Will this be the case no matter with what number we start?

Note.—I am told that this is an old problem, but do not know whether it has ever been solved. (No other number under 100, except, of course, 98, requires so many operations to lead to the desired result, as 89 does.)

SOLUTIONS OF PROBLEMS.

ALGEBRA.

409. Proposed by C. E. GITHINS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelepiped so that its diagonal shall be rational.

(*A*) REMARKS BY ARTEMAS MARTIN, Washington, D. C.

I. On page 162 of the May MONTHLY, Mr. Eells says: "I fail to see, however, how I solved a *different* problem from the one porposed."

The problem proposed is stated above; the problem solved by Mr. Eells follows:

> Find integral values for the edges of a rectangular parallelepiped so that its diagonal, and the diagonal of one of its faces, shall be rational.

The problem proposed does not require the diagonal of one of the faces of the solid to be rational, and the added condition restricts its generality and limits the number of possible solutions.

II. Mr. Eells says on page 269 of the October issue, without any qualifying condition: "This gives the smallest rational parallelepiped," edges 3, 4, 12 and diagonal 13.

III. The fault in my solution was in assuming $x = a$, $y = b$, $d = z + c$. I should have put $x = na$, $y = nb$ and $d = z + nc$; then

$$(na)^2 + (nb)^2 + z^2 = (z + nc)^2 = z^2 + 2ncz + n^2c^2,$$

which gives, after dividing by n,

$$z = \frac{na^2 + nb^2 - nc^2}{2c} \quad \text{and} \quad d = \frac{na^2 + nb^2 + nc^2}{2c}.$$

Now take $n = 2c$ and we have the integral values

$$x = 2ac,\ y = 2bc,\ z = a^2 + b^2 - c^2,\ d = a^2 + b^2 + c^2;$$

therefore

$$(a^2 + b^2 + c^2)^2 = (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2 \qquad (A),$$

whatever be the values of a, b, c; and it is seen, without any elaborate proof, that x and y are always even numbers.

Permuting the letters a, b, c in (A) we get

$$\begin{aligned}(a^2 + b^2 + c^2)^2 &= (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2 \\ &= (a^2 + c^2 - b^2)^2 + (2bc)^2 + (2ab)^2 \\ &= (b^2 + c^2 - a^2)^2 + (2ab)^2 + (2ac)^2.\end{aligned}$$

The additional formulas, however, will only give sets of values for x, y, z that can be obtained by interchanging the numerical values assigned to a, b, c in (A).

In order to obtain values for x, y, z that have no common divisor it is necessary to impose certain restriction on a, b, c. Such values of x, y, z may be called a prime set.

1. The values given to a, b, c must not *contain* a common factor. Hence they can not all be even as in that case x, y, z would be divisible by 2.
2. If a, b, c be all odd, the numbers x, y, z will not have a common divisor.
3. If any two are even and the other odd, the numbers will not have a common divisor.
4. If any two are odd and the other even, then 2 will be even and the numbers will have the common divisor 2 since x and y are always even.
5. But the numbers will be a prime set in all cases after the common factor is divided out.

IV. Mr. Carleton does not put down the rational value of c in his solution on page 164, but it is obvious that

$$c = \sqrt{(2ab)} = \sqrt{(2a \times \tfrac{1}{2}a^3)} = a^2,$$

and that we have

$$(a)^2 + (a^2)^2 + (\tfrac{1}{2}a^3)^2 = a^2 + a^4 + \tfrac{1}{4}a^6 = a(1 + \tfrac{1}{2}a^2)^2,$$

and the numbers (a, b, c) have a common factor a; therefore

$$1^2 + a^2 + \tfrac{1}{4}a^4 = (1 + \tfrac{1}{2}a^2)^2,$$

whatever be the value of a; but, for integers, a must be even.

Fractional values could be avoided by putting $a = 2m^2$, $b = n^2$; for we would then have

$$c = \sqrt{(2ab)} = \sqrt{(4m^2n^2)} = 2mn,$$

and

$$(2m^2)^2 + (2mn)^2 + (n^2)^2 = (2m^2 + n^2)^2,$$

where m and n must be prime to each other.

See the *Mathematical Magazine*, Vol. II., No. 5 (Washington, D. C., Oct., 1891), page 72, where the writer published this method of solution. For other solutions, see the same No., pp. 71–75.

V. It has been claimed that the equation (A) above includes all possible solutions of

$$x^2 + y^2 + z^2 = 0,$$

but the writer is not at present prepared to affirm or deny that claim.

(B) Remarks by J. W. Young, Dartmouth College.

After reading the remarks by W. C. Eells, on Algebra Problem 409, in the April number of the Monthly, it occurred to me that the geometric formulation for the right-triangle problem given by Klein in his *Elementarmath. v. höheren Standp. aus,* Vol. I, should yield a general solution also for the rectangular parallelepiped case. The problem is to solve in integers the equation

$$a^2 + b^2 + c^2 = d^2 \quad (> 0). \tag{1}$$

Placing $x = \frac{a}{d}$, $y = \frac{b}{d}$, $z = \frac{c}{d}$ the equation becomes

$$x^2 + y^2 + z^2 = 1. \tag{2}$$

Equation (1) will be completely solved, if we find all *rational* points on the sphere (2). $(-1, 0, 0)$ is one such point. Any straight line through $(-1, 0, 0)$ and any other rational point will have the equations

$$\frac{x+1}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}, \tag{3}$$

with λ, μ, ν integral and without a common factor; and conversely, every such line (3) will cut out a rational point, P, besides $(-1, 0, 0)$. Solving (2) and (3) we find

$$P = \left(\frac{\lambda^2 - \mu^2 - \nu^2}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\mu\lambda}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\nu\lambda}{\lambda^2 + \mu^2 + \nu^2}\right).$$

If D represents the H. C. F. of $\lambda^2 - \mu^2 - \nu^2$, $2\mu\lambda$, $2\nu\lambda$, and $\lambda^2 + \mu^2 + \nu^2$, then the *general solution* of (1) is given by

$$(a, b, c, d) = \left(\frac{k}{D}(\lambda^2 - \mu^2 - \nu^2), \frac{k}{D}\cdot 2\mu\lambda, \frac{k}{D}\cdot 2\nu\lambda, \frac{k}{D}(\lambda^2 + \mu^2 + \nu^2)\right), \tag{4}$$

where k is any integer.

Y. A. Le Besgue's solutions, to which reference has already been made by Mr. Eells, yield, for $\delta = 0$, the solutions above for $K = D$. If λ, μ, ν are, *e. g.*, relatively prime, no solutions are duplicated by (4). Moreover it is then easy to see that D is equal to the H. C. F. of λ and $\mu^2 + \nu^2$ or twice this H. C. F. The above method will clearly hold for the general equation $\sum_{i=1}^{n} x_i^2 = n^2$.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r} + \frac{b}{r^2} + \frac{a+b}{r^3} + \frac{a+2b}{r^4} + \frac{2a+3b}{r^5} + \cdots,$$

in which a and b are any numbers, and where each numerator after the second is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January (1914) number of the Monthly.

II. Solution by the Proposer.

Let us designate the above numerators as follows:

$$U_1 = a;\ U_2 = b;\ U_3 = a + b;\ U_n = U_{n-1} + U_{n-2}.$$